Constraint Equations for 2D-3D Pose Estimation in Conformal Geometric Algebra

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Abstract

In this paper we present a framework to formalize several different scenarios for pose estimation in a unifying natural algebraic embedding by the use of conformal geometric algebra. The main contribution of the paper is the theoretical analysis and embedding of different pose estimation scenarios. It can be seen that the conformal geometric algebra is more suited to describing the pose estimation scenario, than either the motor algebra or dual quaternion algebra [11]. The different scenarios described below relate projection rays to 3D points, projection planes to 3D points or lines, and projection rays to spheres and circles. The constraints are mostly very compact and easy to implement.

Contents

1 Introduction

In recent years, several mathematical frameworks have been used to describe the 2D-3D pose estimation problem. By pose estimation we mean estimating the transformation (the rigid body motion) between two coordinate frames of measured data and model data [8]. The mathematical frameworks used include matrix and vector algebra, quaternion algebra, dualquaternion algebra and motor algebra [2, 3, 5, 6, 9]. The description in [9] produces compact equations but much care is needed when dealing with signs in the constraint equations - the formulation is far from natural. This is a consequence of the embedding in an even subalgebra where all quantities have to be expressed by scalars, bivectors and pseudoscalars. In this report we try to overcome this problem by using the conformal geometric algebra [4] and we develop constraint equations which contain the same information as those in the motor algebra formalism [9]. However, these are more general and easier to handle, since the constraint equations can then be described by just one product.

The paper is organized as follows. Section 2 gives an introduction to the pose estimation problem. Section 3 introduces the motor algebra and the description of the pose estimation problem in this language. This section is a summary of the algebraic embedding presented in [11]. Section 4 gives an introduction to the conformal geometric algebra and explains some relations to the motor algebra. Furthermore the constraint equations of section 3 are described in the language of conformal geometric algebra and proofs for these connections are given. Two additional constraints are then introduced and analysed. Section 5 ends the paper with a discussion.

2 The scenario of pose estimation

In the scenario of figure 1 we describe the following situation: We assume 3D points $\boldsymbol{Y_i}$, and lines $\boldsymbol{S}_i$ of an object or reference model. Further, we extract line subspaces $\boldsymbol{l}_i$, or points $\boldsymbol{b_i}$ in an image of a calibrated camera, whose optical centre is denoted by $\boldsymbol{c}$, and match them with the model. Three constraints are depicted:

1. **3D point 2D point correspondence:** A transformed point, e.g. $\boldsymbol{X_1}$, of the model point $\boldsymbol{Y_1}$ must lie on the projection ray $\boldsymbol{L}_{b1}$, given by $\boldsymbol{c}$ and the corresponding image point $\boldsymbol{b_1}$.

2. **3D point 2D line correspondence:** A transformed point, e.g. $\boldsymbol{X_1}$, of the model point $\boldsymbol{Y_1}$ must lie on the projection plane $\boldsymbol{P_{12}}$, given by $\boldsymbol{c}$ and the corresponding image line $\boldsymbol{l}_1$.

3. **3D line 2D line correspondence:** A transformed line, e.g. $\boldsymbol{L}_1$, of the model line $\boldsymbol{S}_1$ must lie on the projection plane $\boldsymbol{P_{12}}$, given by $\boldsymbol{c}$ and the corresponding image line $\boldsymbol{l}_1$.

The aim is to embed the scenario in a suitable algebraic language. For this we need a description of the entities, the transformation of the entities and constraints for

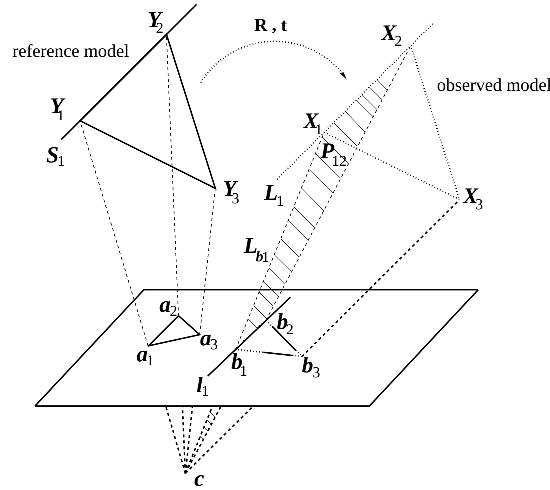

Figure 1: The scenario. The solid lines on the left describe the assumptions: the camera model, the model of the object and the initially extracted lines or points on the image plane. The dashed lines on the right describe the pose of the model, which leads to the best fit of the object with the actual extracted entities.

collinearity and coplanarity of involved entities. Furthermore, these constraints should contain some kind of distance measure, so that they can be used with noisy data resulting from discrete data and measurement errors.

3 Pose estimation in the motor algebra

This section gives an introduction to the motor algebra and the embedding of the pose estimation scenario in the algebraic language of kinematics. A more detailed introduction can be found in [11].

A geometric algebra $\mathcal{G}_{p,q,r}$ is a linear space of dimension 2^n, $n = p + q + r$, with a rich subspace structure. While a vector space has vectors as first order entities, the geometric algebra contains higher order quantities, called blades, with which we can construct multivectors - linear combinations of blades. A geometric algebra $\mathcal{G}_{p,q,r}$ is built from a vector space $\mathbb{R}^n$, endowed with the signature (p, q, r), $n = p + q + r$, by application of a geometric product. The geometric product consists of an outer $(\wedge)$ and an inner $(\cdot)$ product, whose roles are to increase or to decrease the order of the algebraic entities, respectively. For later computations, we will also use the commutator $\underline{\times}$ and anticommutator $\overline{\times}$ product for any two multivectors,

$$\begin{aligned} \boldsymbol{AB} &= \frac{1}{2}(\boldsymbol{AB} + \boldsymbol{BA}) + \frac{1}{2}(\boldsymbol{AB} - \boldsymbol{BA}) \\ &=: \boldsymbol{A}\overline{\times}\boldsymbol{B} + \boldsymbol{A}\underline{\times}\boldsymbol{B}. \end{aligned}$$

For a discussion of these two products and their relation to the geometric, inner and outer product, see [1]. Their role is to separate the symmetric part of the geometric product from the antisymmetric part.

A motor algebra [14] is the 8D even subalgebra $\boldsymbol{\mathcal{G}}^+_{3,0,1}$, derived from $\mathbb{R}^4$, i.e. $n = 4$, $p = 3$, $q = 0$, $r = 1$, with basis vectors γ_k, $k = 1, ..., 4$, and the property $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = +1$ and $\gamma_4^2 = 0$. Because $\gamma_4^2 = 0$, $\boldsymbol{\mathcal{G}}^+_{3,0,1}$ is called a degenerate algebra. The motor algebra $\boldsymbol{\mathcal{G}}^+_{3,0,1}$ is of dimension eight and spanned by qualitatively different subspaces with the following basis multivectors:

one scalar	:	1
six bivectors	:	$\gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2, \gamma_4\gamma_1, \gamma_4\gamma_2, \gamma_4\gamma_3$
one pseudoscalar	:	$\boldsymbol{I} \equiv \gamma_1\gamma_2\gamma_3\gamma_4$.

Because $\gamma_4^2 = 0$, the unit pseudoscalar also squares to zero, i.e. $\boldsymbol{I}^2 = 0$. Since the hypercomplex algebra of quaternions, $\mathbb{H}$, represents a 4D linear space with one scalar and three vector components, it can easily be verified that $\boldsymbol{\mathcal{G}}^+_{3,0,1}$ is isomorphic to the algebra of dual quaternions $\widehat{\mathbb{H}}$ [12]. We will call general elements of the motor algebra motors. The geometric entities of points, lines, and planes have a motor representation. Every motor can be described by two scalars a_0, b_0 and two bivectors $\boldsymbol{a}$, $\boldsymbol{b}$, eg $\boldsymbol{a} = a_1\gamma_{23} + a_2\gamma_{31} + a_3\gamma_{12}$ and the description of the motor is $\boldsymbol{M} = (a_0 + \boldsymbol{a}) + \boldsymbol{I}(b_0 + \boldsymbol{b})$. Changing the sign of the scalar and bivector in the real and the dual parts of the motor leads to the following variants of a motor

$$\boldsymbol{M} = (a_0 + \boldsymbol{a}) + \boldsymbol{I}(b_0 + \boldsymbol{b}) \qquad \widetilde{\boldsymbol{M}} = (a_0 - \boldsymbol{a}) + \boldsymbol{I}(b_0 - \boldsymbol{b})$$

$$\overline{\boldsymbol{M}} = (a_0 + \boldsymbol{a}) - \boldsymbol{I}(b_0 + \boldsymbol{b}) \qquad \widetilde{\overline{\boldsymbol{M}}} = (a_0 - \boldsymbol{a}) - \boldsymbol{I}(b_0 - \boldsymbol{b}) \ .$$

In what follows we will use the term motor in a more restricted sense; we use it to denote a screw transformation. Its constituents are rotation and translation (and dilation in the case of non-unit motors). We represent a rotation by a rotation line axis and a rotation angle. The corresponding entity is called a unit rotor, $\boldsymbol{R}$, and is given by

$$\boldsymbol{R} = r_0 + r_1\gamma_2\gamma_3 + r_2\gamma_3\gamma_1 + r_3\gamma_1\gamma_2 = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{n} = \exp\left(\frac{\theta}{2}\boldsymbol{n}\right).$$

Here θ is the rotation angle and $\boldsymbol{n}$ is the unit orientation vector of the rotation axis, spanned by the bivector basis $\gamma_{12}, \gamma_{31}, \gamma_{12}$.

If on the other hand, $\boldsymbol{t} = t_1\gamma_2\gamma_3 + t_2\gamma_3\gamma_1 + t_3\gamma_1\gamma_2$ is a translation vector in bivector representation, it is represented in the motor algebra by the dual part of a motor, called a translator $\boldsymbol{T}$, where

$$\boldsymbol{T} = 1 + \boldsymbol{I}\frac{\boldsymbol{t}}{2} = \exp\left(\frac{\boldsymbol{t}}{2}\boldsymbol{I}\right).$$

Thus, a translator is also a special kind of rotor.

Because rotation and translation concatenate multiplicatively in the motor algebra, a motor $\mathbf{M}$ reads

$$\boldsymbol{M} = \boldsymbol{T}\boldsymbol{R} = \boldsymbol{R} + \boldsymbol{I}\frac{\boldsymbol{t}}{2}\boldsymbol{R} = \boldsymbol{R} + \boldsymbol{I}\boldsymbol{R}'.$$

A motor represents a general displacement as a screw transformation. For example the line $\boldsymbol{L}$ will be transformed to the line $\boldsymbol{L}'$ by means of a rotation $\boldsymbol{R}_s$ around a line $\boldsymbol{L}_s$ by angle θ, followed by a translation $\boldsymbol{t}_s$ parallel to $\boldsymbol{L}_s$. Then the screw transformation in terms of motors reads

$$\boldsymbol{L}' = \boldsymbol{T}_s\boldsymbol{R}_s\boldsymbol{L}\widetilde{\boldsymbol{R}}_s\widetilde{\boldsymbol{T}}_s = \boldsymbol{M}\boldsymbol{L}\widetilde{\boldsymbol{M}}.$$

constraint	entities	dual quaternion algebra	motor algebra
point-line	point $\boldsymbol{X} = 1 + \boldsymbol{I}\boldsymbol{x}$ line $\boldsymbol{L} = \boldsymbol{n} + \boldsymbol{I}\boldsymbol{m}$	$\boldsymbol{L}\boldsymbol{X} - \boldsymbol{X}\overline{\boldsymbol{L}} = 0$	$\boldsymbol{X}\boldsymbol{L} - \overline{\boldsymbol{L}}\boldsymbol{X} = 0$
point-plane	point $\boldsymbol{X} = 1 + \boldsymbol{I}\boldsymbol{x}$ plane $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{I}d$	$\boldsymbol{P}\overline{\boldsymbol{X}} - \boldsymbol{X}\overline{\boldsymbol{P}} = 0$	$\boldsymbol{P}\boldsymbol{X} - \overline{\boldsymbol{X}\boldsymbol{P}} = 0$
line-plane	line $\boldsymbol{L} = \boldsymbol{n} + \boldsymbol{I}\boldsymbol{m}$ plane $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{I}d$	$\boldsymbol{L}\boldsymbol{P} - \boldsymbol{P}\overline{\boldsymbol{L}} = 0$	$\boldsymbol{L}\boldsymbol{P} + \boldsymbol{P}\overline{\boldsymbol{L}} = 0$

Table 1: The geometric constraints expressed in motor algebra and dual quatenion algebra.

For more detailed discussions see [14] and [13]. We now introduce the description of the most important geometric entities [14].

A point $\boldsymbol{x} \in \mathbb{R}^3$, represented in the bivector basis of $\mathcal{G}^+_{3,0,1}$, i.e. $\boldsymbol{X} \in \mathcal{G}^+_{3,0,1}$, reads $\boldsymbol{X} = 1 + x_1\gamma_4\gamma_1 + x_2\gamma_4\gamma_2 + x_3\gamma_4\gamma_3 = 1 + \boldsymbol{I}\boldsymbol{x}$.

A line $\boldsymbol{L} \in \mathcal{G}^+_{3,0,1}$ is represented by $\boldsymbol{L} = \boldsymbol{n} + \boldsymbol{I}\boldsymbol{m}$ with the line direction $\boldsymbol{n} = n_1\gamma_2\gamma_3 + n_2\gamma_3\gamma_1 + n_3\gamma_1\gamma_2$ and the moment $\boldsymbol{m} = m_1\gamma_2\gamma_3 + m_2\gamma_3\gamma_1 + m_3\gamma_1\gamma_2$.

A plane $\boldsymbol{P} \in \mathcal{G}^+_{3,0,1}$ will be defined by its normal $\boldsymbol{p}$ written as a bivector and by its orthogonal distance to the origin, expressed as the scalar $d = (\boldsymbol{x} \cdot \boldsymbol{p})$, in the following way, $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{I}d$, where $\boldsymbol{x}$ lies on the plane.

In the case of screw motions, $\boldsymbol{M} = \boldsymbol{T}_s\boldsymbol{R}_s$, point and plane transformations can of course be modelled as well as line transformations. These are

$$\boldsymbol{X}' = \boldsymbol{M}\boldsymbol{X}\overline{\widetilde{\boldsymbol{M}}} \qquad \boldsymbol{L}' = \boldsymbol{M}\,\boldsymbol{L}\,\widetilde{\boldsymbol{M}} \qquad \boldsymbol{P}' = \boldsymbol{M}\,\boldsymbol{P}\,\overline{\widetilde{\boldsymbol{M}}}$$

In this study we will use only point and line transformations since points and lines are the entities of our object models.

Now we need only describe the constraints for collinearity and coplanarity. Table 1 gives such an overview of the formulations of the constraints for collinearity and coplanarity in the dual quaternion algebra [7] and motor algebra. We now explicitly evaluate these constraints, in order to simplify the connections and comparisons with the next section.

Evaluating the XL-constraint of a point $\boldsymbol{X} = 1+\boldsymbol{I}\boldsymbol{x}$ collinear with a line $\boldsymbol{L} = \boldsymbol{n}+\boldsymbol{I}\boldsymbol{m}$ leads to

$$\begin{aligned} 0 &= \boldsymbol{X}\boldsymbol{L} - \overline{\boldsymbol{L}}\boldsymbol{X} = (1+\boldsymbol{I}\boldsymbol{x})(\boldsymbol{n}+\boldsymbol{I}\boldsymbol{m}) - (\boldsymbol{n}-\boldsymbol{I}\boldsymbol{m})(1+\boldsymbol{I}\boldsymbol{x}) \\ &= \boldsymbol{n} + \boldsymbol{I}\boldsymbol{m} + \boldsymbol{I}\boldsymbol{x}\boldsymbol{n} - \boldsymbol{n} + \boldsymbol{I}\boldsymbol{m} - \boldsymbol{I}\boldsymbol{n}\boldsymbol{x} = \boldsymbol{I}(2\boldsymbol{m} + \boldsymbol{x}\boldsymbol{n} - \boldsymbol{n}\boldsymbol{x}) \\ &= 2\boldsymbol{I}(\boldsymbol{m} - \boldsymbol{n}\underline{\times}\boldsymbol{x}) \\ \Leftrightarrow 0 &= \boldsymbol{I}(\boldsymbol{m} - \boldsymbol{n}\underline{\times}\boldsymbol{x}). \end{aligned}$$

The term $\boldsymbol{m} - \boldsymbol{n}\underline{\times}\boldsymbol{x}$ means that the moment of a line, which is generated by the outer product of the direction of the line with a point on the line is independent of the chosen point of the line, which is a clear fact from Plücker representation of lines [7].

Evaluating the XP-constraint of a point $\boldsymbol{X} = 1+\boldsymbol{I}\boldsymbol{x}$ coplanar to a plane $\boldsymbol{P} = \boldsymbol{p}+\boldsymbol{I}d$ leads to

$$0 = \boldsymbol{P}\boldsymbol{X} - \overline{\boldsymbol{X}\boldsymbol{P}} = (\boldsymbol{p}+\boldsymbol{I}d)(1+\boldsymbol{I}\boldsymbol{x}) - (1-\boldsymbol{I}\boldsymbol{x})(\boldsymbol{p}-\boldsymbol{I}d)$$

$$
\begin{aligned}
&= \boldsymbol{p} + \boldsymbol{I}\boldsymbol{p}\boldsymbol{x} + \boldsymbol{I}d - \boldsymbol{p} + \boldsymbol{I}d + \boldsymbol{I}\boldsymbol{x}\boldsymbol{p} = \boldsymbol{I}(2d + \boldsymbol{p}\boldsymbol{x} + \boldsymbol{x}\boldsymbol{p}) \\
\Leftrightarrow 0 &= \boldsymbol{I}(d + \boldsymbol{p}\overline{\times}\boldsymbol{x}).
\end{aligned}
$$

The term $d + \boldsymbol{p}\overline{\times}\boldsymbol{x}$ describes the perpendicular distance of the point to the plane [11].

Evaluating the LP-constraint of a line $\boldsymbol{L} = \boldsymbol{n} + \boldsymbol{I}\boldsymbol{m}$ coplanar to a plane $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{I}d$ leads to

$$
\begin{aligned}
0 &= \boldsymbol{L}\boldsymbol{P} + \boldsymbol{P}\overline{\boldsymbol{L}} = (\boldsymbol{n} + \boldsymbol{I}\boldsymbol{m})(\boldsymbol{p} + \boldsymbol{I}d) + (\boldsymbol{p} + \boldsymbol{I}d)(\boldsymbol{n} - \boldsymbol{I}\boldsymbol{m}) \\
&= \boldsymbol{n}\boldsymbol{p} + \boldsymbol{I}\boldsymbol{m}\boldsymbol{p} + \boldsymbol{I}\boldsymbol{n}d + \boldsymbol{p}\boldsymbol{n} + \boldsymbol{I}\boldsymbol{n}d - \boldsymbol{I}\boldsymbol{p}\boldsymbol{m} \\
&= \boldsymbol{n}\boldsymbol{p} + \boldsymbol{p}\boldsymbol{n} + \boldsymbol{I}(2d\boldsymbol{n} - \boldsymbol{p}\boldsymbol{m} + \boldsymbol{m}\boldsymbol{p}) \\
\Leftrightarrow 0 &= \boldsymbol{n}\overline{\times}\boldsymbol{p} + \boldsymbol{I}(d\boldsymbol{n} - \boldsymbol{p}\underline{\times}\boldsymbol{m})
\end{aligned}
$$

Thus, the constraint of coplanarity of a line to a plane can be partitioned into a constraint on the real part of the motor and a constraint on the dual part of the motor. The constraint on the real part describes the angle between the direction of the line and the normal to the plane and the dual part contains some distance measure between the line and the plane. Indeed it can be shown, that for $\boldsymbol{n} \perp \boldsymbol{p}$, $d\boldsymbol{n} - \boldsymbol{p}\underline{\times}\boldsymbol{m}$ reduces to the XP constraint, for a point on the line [11].

Using the point $\boldsymbol{Y}$ and line $\boldsymbol{S}$ from section two, the constraints for pose estimation read

$$
\begin{aligned}
(\boldsymbol{M}\boldsymbol{Y}\widetilde{\overline{\boldsymbol{M}}})\boldsymbol{L} - \overline{\boldsymbol{L}}(\boldsymbol{M}\boldsymbol{Y}\widetilde{\overline{\boldsymbol{M}}}) &= 0 \\
\boldsymbol{P}(\boldsymbol{M}\boldsymbol{Y}\widetilde{\overline{\boldsymbol{M}}}) - \overline{(\boldsymbol{M}\boldsymbol{Y}\widetilde{\overline{\boldsymbol{M}}})}\overline{\boldsymbol{P}} &= 0 \\
(\boldsymbol{M}\boldsymbol{S}\widetilde{\boldsymbol{M}})\boldsymbol{P} + \boldsymbol{P}\overline{(\boldsymbol{M}\boldsymbol{S}\widetilde{\boldsymbol{M}})} &= 0.
\end{aligned}
$$

The solution to these equations will satisfy the pose estimation problem at hand: find the best motor $\boldsymbol{M}$ which satisfies the constraints. The constraints have the interesting property that the perpendicular distances are the natural distance measures. They are extremely useful in applications in digital image processing where we are dealing with noisy data [9].

4 Pose estimation in conformal geometric algebra

In this section, we will first give an introduction to the construction of the conformal geometric algebra (ConfGA) and introduce the entities we use in this context. A more detailed introduction can be found in [4]. Then we describe the connection of the description of the entities in ConfGA to their corresponding description in the motor algebra and estimate the constraints of the previous section in this algebra.

4.1 Introduction to conformal geometric algebra

To introduce the ConfGA, we follow [4] and start with the *Minkowski plane* $\mathcal{G}_{1,1,0}$, which has an orthonormal basis $\{\mathbf{e}_+, \mathbf{e}_-\}$, defined by the properties

$$\mathrm{e}_+^2 = 1, \quad \mathrm{e}_-^2 = -1 \text{ and } \quad \mathrm{e}_+ \cdot \mathrm{e}_- = 0.$$

A *Null basis* can now be introduced by the vectors

$$\mathrm{e}_0 = \tfrac{1}{2}(\mathrm{e}_- - \mathrm{e}_+) \text{ and } \quad \mathrm{e} = \mathrm{e}_- + \mathrm{e}_+.$$

Furthermore we define

$$\boldsymbol{E} := \mathrm{e} \wedge \mathrm{e}_0.$$

For these additional elements the following properties are straightforward to prove and can be summarized as

$$\begin{array}{lll} \mathrm{e}_0^2 = \mathrm{e}^2 = 0, & \mathrm{e} \cdot \mathrm{e}_0 = -1\ , & \boldsymbol{E} = \mathrm{e}_+\mathrm{e}_- \\ \boldsymbol{E}^2 = 1, & \boldsymbol{E}\mathrm{e} = -\mathrm{e}\boldsymbol{E} = -\mathrm{e}\ , & \boldsymbol{E}\mathrm{e}_0 = -\mathrm{e}_0\boldsymbol{E} = \mathrm{e}_0. \end{array}$$

These properties will be used in the following computations. In an n-dimensional vector space, the Minkowski model $\mathcal{G}_{n+1,1,0}$ will be used, therefore enlarging the Geometric Algebra of the n-dimensional vector space by two additional basis vectors, which define a *Null space*. For the interpretation of the additional basis vectors, it helps to identify e_0 with the origin, and e with the point at infinity. This is also explained in [4].

The set of all null vectors is called the *null cone* and the surface

$$\{\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1,0} | \underline{\boldsymbol{x}}^2 = 0, \underline{\boldsymbol{x}} \cdot \mathrm{e} = -1\}$$

is termed the *horosphere*: there is a 1-1 mapping between points in $\mathcal{G}_{n,0,0}$ and the horosphere. The general form of the points $\boldsymbol{x} \in \mathcal{G}_{n,0,0}$ can also be described by

$$\underline{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^2\mathrm{e} + \mathrm{e}_0.$$

Lines can be described by the outer product of two points on the line and the point at infinity

$$\underline{\boldsymbol{L}} \;=\; \mathrm{e} \wedge \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}}$$

Since the outer product of three points determines a circle [4], the line can be interpreted as a circle passing through the point at infinity.

Planes can be described by the outer product of three points on the plane, and the point at infinity

$$\underline{\boldsymbol{P}} \;=\; \mathrm{e} \wedge \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{c}}$$

Using $\mathrm{e} \wedge \underline{\boldsymbol{a}}$ instead of $\underline{\boldsymbol{a}}$ (this is the so called *affine* representation of a point [4]), we can write the point, line and plane as

$$\begin{array}{llll} \underline{\boldsymbol{X}} = \mathrm{e} \wedge \underline{\boldsymbol{x}} & = & \boldsymbol{E} + \mathrm{e}\boldsymbol{x} & \\ \underline{\boldsymbol{L}} = \mathrm{e} \wedge \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} & = & \boldsymbol{E}(\boldsymbol{b} - \boldsymbol{a}) + \mathrm{e}\boldsymbol{a} \wedge \boldsymbol{b} & = \boldsymbol{E}\boldsymbol{n} + \mathrm{e}\boldsymbol{M} \\ \underline{\boldsymbol{P}} = \mathrm{e} \wedge \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{c}} & = & \boldsymbol{E}(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a}) + \mathrm{e}\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} & = \boldsymbol{E}\boldsymbol{P} + \mathrm{e}d\boldsymbol{I}. \end{array}$$

Note that $\underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} = \underline{\boldsymbol{a}} \wedge (\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}})$, and $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = d\boldsymbol{I}$, with $\boldsymbol{I} = \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3, d \in \mathbb{R}$. It is now easy to see the similarities between these representations and those in the motor algebra.

Notice however, that the coefficients of the entities remain in the natural environment, so a vector is denoted by a vector, a bivector by a bivector, etc and no codings, such as those used in the motor algebra or dual quaternion algebra approaches, are necessary.

As in the previous section, rotations can be described by rotors $\boldsymbol{R}$. If $\boldsymbol{x} \to \boldsymbol{R}\boldsymbol{x}\widetilde{\boldsymbol{R}}$, and $\underline{\boldsymbol{x}} = F(\boldsymbol{x})$ then we find that we get the rotated vector in 5D by mapping the rotated 3D vector

$$\begin{aligned} \boldsymbol{R}F(\boldsymbol{x})\widetilde{\boldsymbol{R}} = \boldsymbol{R}\underline{\boldsymbol{x}}\widetilde{\boldsymbol{R}} &= \boldsymbol{R}\boldsymbol{x}\widetilde{\boldsymbol{R}} + \frac{1}{2}\boldsymbol{x}^2\boldsymbol{R}\mathrm{e}\widetilde{\boldsymbol{R}} + \boldsymbol{R}\mathrm{e}_0\widetilde{\boldsymbol{R}} \\ &= \boldsymbol{R}\boldsymbol{x}\widetilde{\boldsymbol{R}} + \frac{1}{2}\boldsymbol{x}^2\boldsymbol{R}\widetilde{\boldsymbol{R}}\mathrm{e} + \boldsymbol{R}\widetilde{\boldsymbol{R}}\mathrm{e}_0 \\ &= \boldsymbol{R}\boldsymbol{x}\widetilde{\boldsymbol{R}} + \frac{1}{2}\boldsymbol{x}^2\mathrm{e} + \mathrm{e}_0 = F(\boldsymbol{R}\boldsymbol{x}\widetilde{\boldsymbol{R}}) \end{aligned}$$

A translation can be described by a translator $\boldsymbol{T_a} = (1+\frac{\boldsymbol{a}}{2})\mathrm{e}_0$ (also similarly to section 3) and is nothing more than a special rotor. Indeed it can be shown [15], that translations, rotations, dilations and inversions can all be described by suitable rotors in ConfGA.

4.2 Formalization of the constraints in conformal geometric algebra

In this section we will estimate the constraints for collinearity and coplanarity in ConfGA. As a result we will see that the equations contain exactly the same information as in the motor algebra approach, but are expressed more easily.

Evaluating the commutator product of a point $\underline{\boldsymbol{X}}$ and a line $\underline{\boldsymbol{L}}$ leads to

$$\begin{aligned} \underline{\boldsymbol{X}} \underline{\times} \underline{\boldsymbol{L}} &= \frac{1}{2}(\underline{\boldsymbol{X}}\underline{\boldsymbol{L}} - \underline{\boldsymbol{L}}\underline{\boldsymbol{X}}) \\ &= \frac{1}{2}((\boldsymbol{E} + \mathrm{e}\boldsymbol{x})(\boldsymbol{E}\boldsymbol{n} + \mathrm{e}\boldsymbol{M}) - (\boldsymbol{E}\boldsymbol{n} + \mathrm{e}\boldsymbol{M})(\boldsymbol{E} + \mathrm{e}\boldsymbol{x})) \\ &= \frac{1}{2}(\mathrm{e}\boldsymbol{x}\boldsymbol{E}\boldsymbol{n} + \boldsymbol{E}\mathrm{e}\boldsymbol{M} + \boldsymbol{E}^2\boldsymbol{n} - \mathrm{e}\boldsymbol{M}\boldsymbol{E} - \boldsymbol{n}\boldsymbol{E}\mathrm{e}\boldsymbol{x} - \boldsymbol{n}) \\ &= \frac{1}{2}(\mathrm{e}\boldsymbol{x}\boldsymbol{n} - \boldsymbol{M}\mathrm{e} - \boldsymbol{M}\mathrm{e} - \boldsymbol{n}\boldsymbol{x}\mathrm{e}) \\ &= \frac{1}{2}(-(2\boldsymbol{M} - (\boldsymbol{x}\boldsymbol{n} - \boldsymbol{n}\boldsymbol{x}))\mathrm{e}) \\ &= -(\boldsymbol{M} - (\boldsymbol{x} \underline{\times} \boldsymbol{n}))\mathrm{e} \end{aligned}$$

Comparing the resulting term with the term evaluated from the corresponding XL-motor algebra constraint, it is easy to see, that (neglecting the sign) the two forms have the same content. Note the change of the sign, occuring from bivectors in the motor algebra with respect to the vectors we have in the ConfGA.

Evaluating the commutator product of a point $\underline{\boldsymbol{X}}$ and a plane $\underline{\boldsymbol{P}}$ leads to

$$\underline{\boldsymbol{X}} \underline{\times} \underline{\boldsymbol{P}} = \frac{1}{2}(\underline{\boldsymbol{X}}\underline{\boldsymbol{P}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{X}})$$

$$
\begin{aligned}
&= (\boldsymbol{E}+\mathrm{e}\boldsymbol{x})(\boldsymbol{E}\boldsymbol{P}+\mathrm{e}d\boldsymbol{I})-(\boldsymbol{E}\boldsymbol{P}+\mathrm{e}d\boldsymbol{I})(\boldsymbol{E}+\mathrm{e}\boldsymbol{x})\\
&= \frac{1}{2}(\mathrm{e}\boldsymbol{x}\boldsymbol{E}\boldsymbol{P}+\boldsymbol{E}\mathrm{e}d\boldsymbol{I}+\boldsymbol{P}-\mathrm{e}d\boldsymbol{I}\boldsymbol{E}-\boldsymbol{E}\boldsymbol{P}\mathrm{e}\boldsymbol{x}-\boldsymbol{P})\\
&= \frac{1}{2}(-\boldsymbol{x}\boldsymbol{P}\mathrm{e}+d\boldsymbol{I}\mathrm{e}+\boldsymbol{P}+d\boldsymbol{I}\mathrm{e}-\boldsymbol{P}\boldsymbol{x}\mathrm{e}-\boldsymbol{P})\\
&= \frac{1}{2}(2d\boldsymbol{I}-(\boldsymbol{x}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{x}))\mathrm{e}\\
&= (d\boldsymbol{I}-(\boldsymbol{x}\overline{\times}\boldsymbol{P}))\mathrm{e}
\end{aligned}
$$

Note here that the anticommutator product of a bivector and a vector leads to a trivector, which is subtracted from $d\boldsymbol{I}$. Comparing the resulting term with the term evaluated from the corresponding XP motor algebra expression, it is easy to see that again the two have the same content.

Evaluating the anticommutator product of a line $\underline{\boldsymbol{L}}$ and a plane $\underline{\boldsymbol{P}}$ leads to

$$
\begin{aligned}
\underline{\boldsymbol{L}}\overline{\times}\underline{\boldsymbol{P}} &= \frac{1}{2}(\underline{\boldsymbol{L}}\underline{\boldsymbol{P}}+\underline{\boldsymbol{P}}\underline{\boldsymbol{L}})\\
&= \frac{1}{2}((\boldsymbol{E}\boldsymbol{n}+\mathrm{e}\boldsymbol{M})(\boldsymbol{E}\boldsymbol{P}+\mathrm{e}d\boldsymbol{I})+(\boldsymbol{E}\boldsymbol{P}+\mathrm{e}d\boldsymbol{I})(\boldsymbol{E}\boldsymbol{n}+\mathrm{e}\boldsymbol{M}))\\
&= \frac{1}{2}(\mathrm{e}\boldsymbol{M}\boldsymbol{E}\boldsymbol{P}+\boldsymbol{n}\boldsymbol{E}\mathrm{e}\boldsymbol{I}d+\boldsymbol{n}\boldsymbol{P}+\mathrm{e}\boldsymbol{I}d\boldsymbol{n}\boldsymbol{E}+\boldsymbol{E}\boldsymbol{P}\mathrm{e}\boldsymbol{M}+\boldsymbol{E}\boldsymbol{P}\boldsymbol{n}\boldsymbol{E})\\
&= \frac{1}{2}(\boldsymbol{M}\boldsymbol{P}\mathrm{e}+\boldsymbol{n}\boldsymbol{I}d\mathrm{e}+\boldsymbol{n}\boldsymbol{P}+\boldsymbol{I}d\boldsymbol{n}\mathrm{e}-\boldsymbol{P}\boldsymbol{M}\mathrm{e}+\boldsymbol{P}\boldsymbol{n})\\
&= \frac{1}{2}((\boldsymbol{n}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{n})+(2\boldsymbol{n}\boldsymbol{I}d+\boldsymbol{M}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{M})\mathrm{e})\\
&= \frac{1}{2}((\boldsymbol{n}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{n})+2(\boldsymbol{n}\boldsymbol{I}d+\boldsymbol{M}\underline{\times}\boldsymbol{P})\mathrm{e})\\
&= \boldsymbol{n}\overline{\times}\boldsymbol{P}+(\boldsymbol{n}\boldsymbol{I}d+\boldsymbol{M}\underline{\times}\boldsymbol{P})\mathrm{e}
\end{aligned}
$$

Note that the commutator product of two bivectors leads to a bivector which is added to $\boldsymbol{n}\boldsymbol{I}d$. Comparing the resulting term with the term evaluated from the corresponding LP motor algebra expression, we see once again that the two expressions have the same content.

Thus, the pose estimation constraint equations reduce to setting the commutator and anticommutator products to zero and the problem of pose estimation can be reduced to finding the best rotor $\boldsymbol{R}$, meaning rotation and translation, which satisfies

$$
\begin{aligned}
(\boldsymbol{R}\underline{\boldsymbol{X}}\tilde{\boldsymbol{R}})\underline{\times}\underline{\boldsymbol{L}} &= 0\\
(\boldsymbol{R}\underline{\boldsymbol{X}}\tilde{\boldsymbol{R}})\underline{\times}\underline{\boldsymbol{P}} &= 0\\
(\boldsymbol{R}\underline{\boldsymbol{L}}\tilde{\boldsymbol{R}})\overline{\times}\underline{\boldsymbol{P}} &= 0
\end{aligned}
$$

Note that the representation of the entities are in general not scaled as they are in the motor algebra (eg the direction of a line $(\boldsymbol{b}-\boldsymbol{a})$ is in general not a unit vector). The constraints are therefore equivalent up to a scaling factor which can easily be estimated.

In terms of the outer and inner products it can also be shown

$$\begin{aligned} ((\boldsymbol{R}\underline{\boldsymbol{X}}\tilde{\boldsymbol{R}})\underline{\times}\underline{\boldsymbol{L}}) \cdot \mathrm{e}_0 &= -((\boldsymbol{R}\underline{\boldsymbol{x}}\tilde{\boldsymbol{R}}) \wedge \underline{\boldsymbol{L}}) \cdot \boldsymbol{E} \\ ((\boldsymbol{R}\underline{\boldsymbol{X}}\tilde{\boldsymbol{R}})\underline{\times}\underline{\boldsymbol{P}}) \cdot \mathrm{e}_0 &= ((\boldsymbol{R}\underline{\boldsymbol{x}}\tilde{\boldsymbol{R}}) \wedge \underline{\boldsymbol{P}}) \cdot \boldsymbol{E} \\ ((\boldsymbol{R}\underline{\boldsymbol{L}}\tilde{\boldsymbol{R}})\overline{\times}\underline{\boldsymbol{P}}) \cdot \boldsymbol{I}_5 &= (\boldsymbol{R}\underline{\boldsymbol{L}}\tilde{\boldsymbol{R}}) \cdot (\underline{\boldsymbol{P}}\boldsymbol{I}_5), \end{aligned}$$

with $\underline{\boldsymbol{X}} = \mathrm{e} \wedge \underline{\boldsymbol{x}}$, and $\boldsymbol{I}_5 = \mathrm{e}_-\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_+$. Using these formulations, it is easy to see that the commutator and anticommutator products can also be replaced by suitable inner and outer products. This is not possible in the motor algebra.

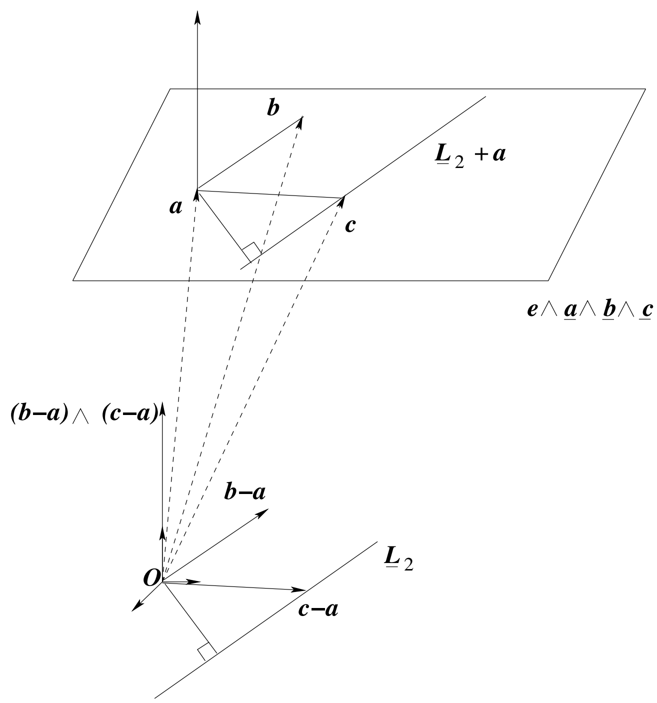

Figure 2: Relation of the XL- constraint to the plane representation: the normal of the plane leads to an intrinsic coding of the distance from a point to a line.

It is interesting to mention the following: a plane can be described by three points on the plane, eg $\underline{\boldsymbol{a}}$, $\underline{\boldsymbol{b}}$, $\underline{\boldsymbol{c}}$ leading to $\underline{\boldsymbol{P}} = \mathrm{e} \wedge \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{c}}$. This representation consists of the volume $\mathbf{e}\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}$, and the normal $\boldsymbol{E}(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a})$. The normal can be interpreted as the moment of the line $\underline{\boldsymbol{L}}_2$ given by the direction $(\boldsymbol{b} - \boldsymbol{a})$ and passing through the point $(\boldsymbol{c} - \boldsymbol{a})$. Since the magnitude of the moment describes the orthogonal distance of the origin to a line, the magnitude of the normal describes exactly the orthogonal distance of the point $\boldsymbol{a}$ to the line $\underline{\boldsymbol{L}}_2$ parallel-transported to pass through the point $\boldsymbol{c}$.

Thus, within the normal of the plane we have an intrinsic coding of the distance from a point to a line. Figure 2 visualizes this geometry.

We note here that the XP constraint can further be interpreted as a one dimensional extension of the XL constraint, and the LP constraint uses the dual representation of a plane, and the inner product, to relate a line to a plane. Since the information is coded in different blades of $\boldsymbol{\mathcal{G}}_{n+1,1,0}$ we require these different products between the terms.

Compared with the formulations in the motor algebra, the constraints are much more compact and describe the constraints using only one product (commutator, anticommutator or outer and inner product). As described for the motor algebra the information obtained in such a formulation will be close to optimal for certain problems.

4.3 Spheres and circles in conformal geometric algebra

In this section we will construct constraints for a projection ray being tangential to a 3D sphere or 3D circle. Since 3D spheres are more easily described in 3D space than circles, we will start with spheres, and then continue with circles.

As explained in [4], spheres can be described by four points lying on the sphere, $\Omega = \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{c}} \wedge \underline{\boldsymbol{d}}$. Let this sphere have radius ρ and centre $\underline{\boldsymbol{m}}$. The distance of a projection ray $\underline{\boldsymbol{L}}$ to the sphere Ω can be described by the distance of the projection ray $\underline{\boldsymbol{L}}$ to the centre $\underline{\boldsymbol{m}}$ minus the radius ρ. Here the $\dagger$ denotes tangentiality:

$$\begin{aligned} \underline{\boldsymbol{L}} &\dagger \Omega \\ \Leftrightarrow dist(\underline{\boldsymbol{L}}, \underline{\boldsymbol{m}}) &= \rho \end{aligned}$$

The distance of a line to a point can be described by the commutator product, but indeed it is *not* possible to write $dist(\underline{\boldsymbol{L}}, \underline{\boldsymbol{m}}) = (\underline{\boldsymbol{L}} \underline{\times} \underline{\boldsymbol{m}})^2$, since the resulting error vector is coded in the Nullspace of $\boldsymbol{\mathcal{G}}_{n+1,1,0}$, ie $(\underline{\boldsymbol{L}} \underline{\times} \underline{\boldsymbol{m}})^2 = 0$.

One of the nice features of ConfGA is that it is possible to change between the Nullspace and the non-Nullspace, in this case by taking the inner product with $\mathbf{e}_0$ since $\mathbf{e} \cdot \mathbf{e}_0 = -1$, a fact which is extremely useful and not possible in the motor algebra. In the motor algebra the problem was solved by simply neglecting the pseudoscalar $\boldsymbol{I}$. The distance between a line and a sphere can then be described directly by

$$\begin{aligned} dist(\underline{\boldsymbol{L}}, \underline{\boldsymbol{m}}) &= \rho \\ \Leftrightarrow \sqrt{((\underline{\boldsymbol{L}} \underline{\times} \underline{\boldsymbol{m}}) \cdot \mathbf{e}_0)^2} - \rho &= 0 \\ \Leftrightarrow \|(\underline{\boldsymbol{L}} \underline{\times} \underline{\boldsymbol{m}}) \cdot \mathbf{e}_0\| - \rho &= 0 \end{aligned}$$

This describes, in the context of pose estimation, an interesting extension, since now it is also possible to estimate the relative position of a ball, or a sphere in the space.

Much harder are circles in 3D space. Circles can be described by three points lying on the circle: $\omega = \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{c}}$. The general expression leads to

$$\begin{aligned} \omega &= \underline{\boldsymbol{a}} \wedge \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{c}} \\ &= A + A^{-}\mathbf{e} + A^{+}\mathbf{e}_0 + A^{+-}\boldsymbol{E} \end{aligned}$$

with

$$\begin{aligned} A &= \boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} & A^{-} &= \tfrac{1}{2}(\boldsymbol{c}^2(\boldsymbol{a} \wedge \boldsymbol{b}) - \boldsymbol{b}^2(\boldsymbol{a} \wedge \boldsymbol{c}) + \boldsymbol{a}^2(\boldsymbol{b} \wedge \boldsymbol{c})) \\ A^{+} &= \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{b} \wedge \boldsymbol{c} - \boldsymbol{a} \wedge \boldsymbol{c} & A^{+-} &= \tfrac{1}{2}(\boldsymbol{a}(\boldsymbol{b}^2 - \boldsymbol{c}^2) + \boldsymbol{b}(\boldsymbol{c}^2 - \boldsymbol{a}^2) + \boldsymbol{c}(\boldsymbol{a}^2 - \boldsymbol{b}^2)) \end{aligned}$$

One way of describing the circle geometrically is by the plane in which the circle lies ($\mathbf{e} \wedge \omega$ gives the plane) and the line which is perpendicular to the plane and meets it at the centre of the circle (A^{+-} contains the moment of the line). It can be shown that the radius ρ can be described by

$$\rho^2 \quad = \quad -\frac{\omega^2}{(\mathbf{e}\wedge\omega)^2}$$

and the centre is given by

$$\boldsymbol{M}_\omega = -\rho^2\left(\frac{(\mathbf{e}\wedge\omega)\omega}{\omega^2}+\frac{1}{2}\mathbf{e}\right) \quad \text{or} \quad \underline{\boldsymbol{M}}_\omega = \mathbf{e}\wedge\boldsymbol{M}_\omega$$

The aim is to formalize a constraint equation which leads to the orthogonal distance of the projection ray $\underline{\boldsymbol{L}}$ to the circle ω.

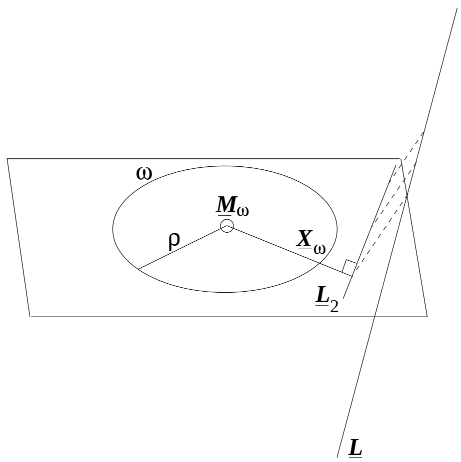

Figure 3: The orthogonal distance of the projection ray $\underline{\boldsymbol{L}}$ to the circle ω, is obtained by estimating the nearest point, $\underline{\boldsymbol{X}}_\omega$, of the circle ω to the line $\underline{\boldsymbol{L}}$ and evaluating this error measure.

One way to do this is to estimate the nearest point $\underline{\boldsymbol{X}}_\omega$ of the circle ω to the line $\underline{\boldsymbol{L}}$ and evaluate this error measure, see figure 3. To estimate the point $\underline{\boldsymbol{X}}_\omega$, the key ideas are to make an orthographic projection of the line $\underline{\boldsymbol{L}}$ to the plane $\underline{\boldsymbol{P}}_1 := \mathbf{e}\wedge\omega$, to estimate the shortest distance from the centre of the circle to the resulting line, and to establish the point $\underline{\boldsymbol{X}}_\omega$ from this information. Suppose the following are given: the circle ω, the projection ray $\underline{\boldsymbol{L}}$, the center $\underline{\boldsymbol{M}}_\omega$ or $\boldsymbol{M}_\omega$ of the circle, the radius ρ of the circle, the normal $\underline{\boldsymbol{T}}$ of the plane in which the circle lies and the direction $\boldsymbol{n}$ of the projection ray $\underline{\boldsymbol{L}}$. These quantities are simple to extract from a given circle ω and projection ray $\underline{\boldsymbol{L}}$. The orthogonal distance, of the circle ω to the line $\underline{\boldsymbol{L}}$ can now be established by the following steps:

1. Orthographic projection of the line $\underline{\boldsymbol{L}}$ to the plane $\underline{\boldsymbol{P}}_1 := \mathbf{e}\wedge\omega$, leads to a line $\underline{\boldsymbol{L}}_2$:

$$\begin{aligned}\underline{\boldsymbol{P}}_2 &:= \underline{\boldsymbol{L}}\wedge\underline{\boldsymbol{T}}\\ \underline{\boldsymbol{L}}_2 &:= (\underline{\boldsymbol{P}}_1\vee\underline{\boldsymbol{P}}_2) = -(\underline{\boldsymbol{P}}_1\underline{\times}\underline{\boldsymbol{P}}_2)\boldsymbol{I}\boldsymbol{E}\end{aligned}$$

2. The error vector from the centre of the circle $\boldsymbol{M}_\omega$ to the line $\underline{\boldsymbol{L}}_2$, with unit normal $\boldsymbol{n_2}$, can be estimated via

$$\begin{aligned} \boldsymbol{X}_{d1} &:= ((\underline{\boldsymbol{M}_\omega \times \boldsymbol{L}_2}) \cdot \mathbf{e}_0) \cdot \boldsymbol{n_2} \\ \boldsymbol{X}_{d2} &:= \frac{\mathbf{e} \wedge \boldsymbol{X}_{d1}}{\|\boldsymbol{X}_{d1}\|} \end{aligned}$$

3. The nearest point to the line can be established by

$$\underline{\boldsymbol{X}}_\omega := \underline{\boldsymbol{M}}_\omega + \rho \boldsymbol{X}_{d2}$$

4. The error can now be described by the following equation as in the previous section

$$(\underline{\boldsymbol{X}_\omega \times \boldsymbol{L}}) = 0.$$

Though the steps themselves are very easy to implement, the substitution to the general form of the constraint is no longer simple and it is no longer straightforward to apply general transformations in the form of rotors. Future work will look at finding a neater, possibly closed-form version of the constraint.

5 Discussion

In this paper we present a framework to formalize several different scenarios for pose estimation in a unifying natural algebraic embedding by the use of conformal geometric algebra. The main contribution of the paper is the theoretical analysis and embedding of different pose estimation scenarios, and we hope to have shown that the conformal geometric algebra is much more suitable in describing this scenario, than either motor algebra or the dual quaternion algebra [11]. Since the evaluation of the XL, XP, LP constraints leads to exactly the same information as in the motor algebra approach, no new experiments are presented. Two additional constraints relating projection rays to 3D circles or spheres have not been implemented and such experiments are the task of future work. Since the entities and their transformations are more naturally represented in conformal geometric algebra, it will also be interesting, as part of future work, to analyse existing formalizations of the dual quaternion algebra in conformal geometric algebra.

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